MODERN COSMOLOGY

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Lecture 1

General information and basic parameters of the Universe– 1

- Cosmology - the science about evolution of the Universe
- Units of measurement

  1 pc ~ 3 light-year

  1 light-year ~ 10^{18} sm

  Solar mass $M_{sol} = 2 \cdot 10^{33}$ g

  Sun luminosity $L_{sol} = 4 \cdot 10^{33}$ erg/s

- Planck units ($h = c = 1$)

  Planck mass $M_{Pl} \sim 10^{-5}$ г $\sim 10^{19}$ GeV

  Planck length $l_{Pl} \sim 10^{-33}$ sm

  Plack time $t_{Pl} \sim 10^{-44}$ s

General information and basic parameters of the Universe– 2

- Parameters of our Universe

  Size of the vizible part ~ 10^{28} sm ~ $6{\cdot}10^{3}$ Mpc

  Lifetime of the Universe~ 13.7 billion years

- Components of the Universe

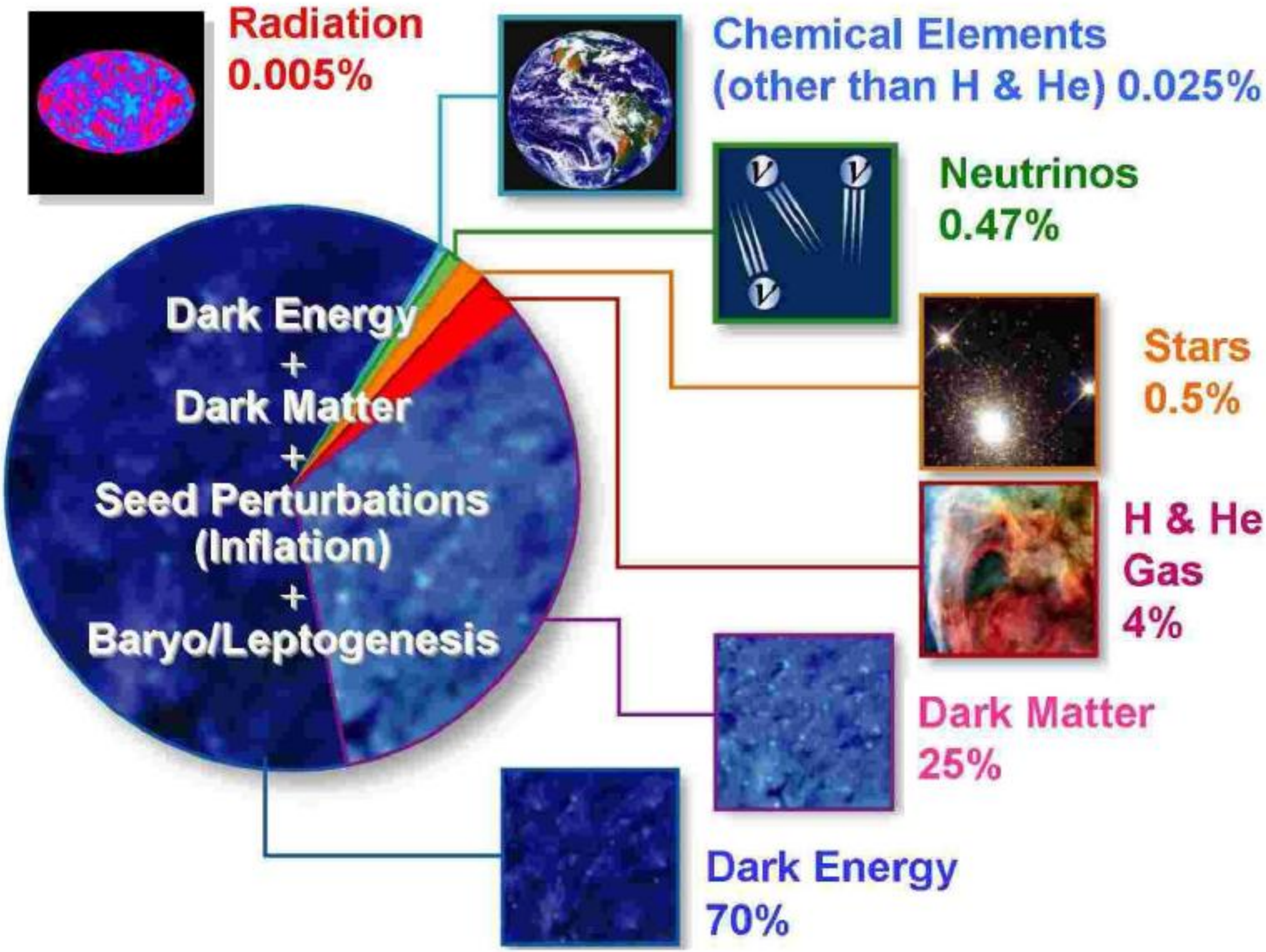
  Dark Energy $\Omega_\Lambda \approx 0.72$

  Hidden mass (dark matter) $\Omega_{dm} \approx 0.24$

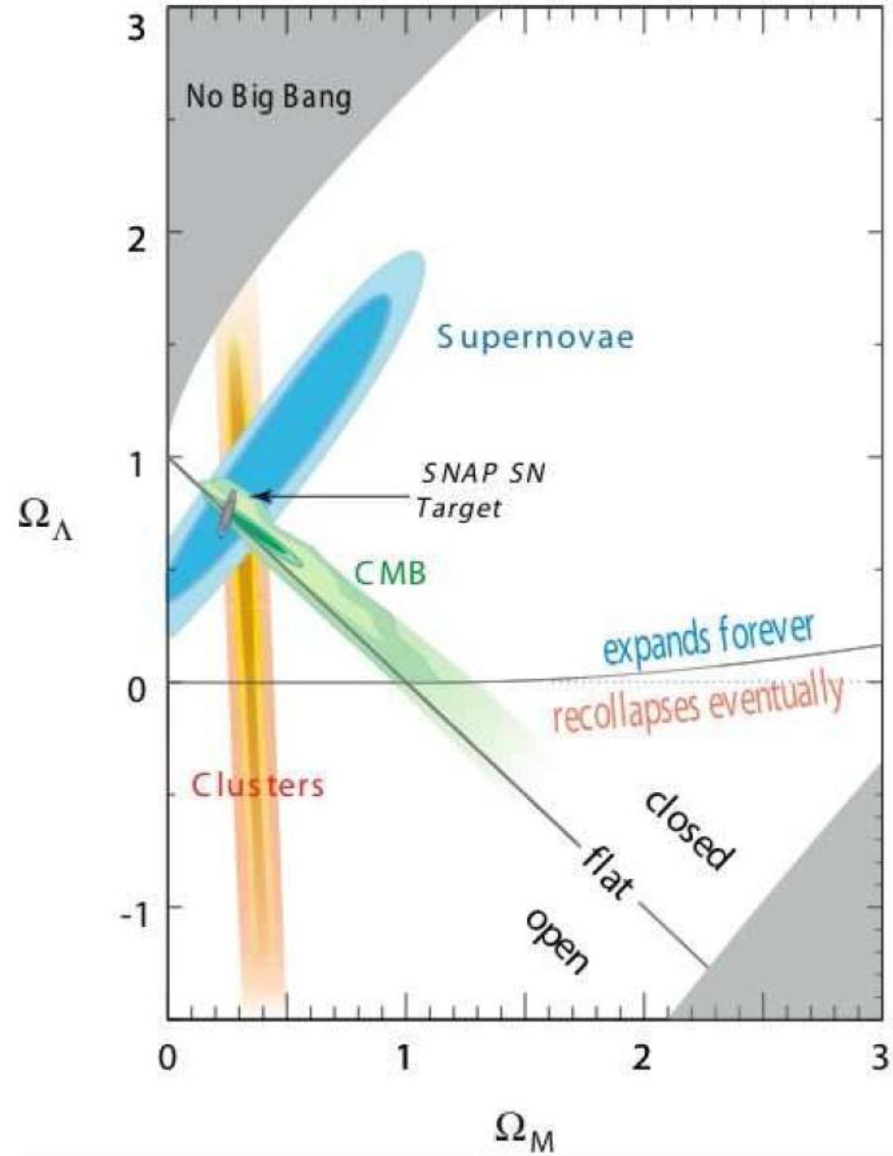
  Baryon matter $\Omega_b \approx 0.04$

  ~ 10^{11} of galaxies, concentrated in clusters with voids

Problem – How many nucleons does Universe contain?



Radiation
0.005%
Chemical Elements
(other than H & He) 0.025%
Neutrinos
0.47%
Stars
0.5%
H & He
Gas
4%
Dark Matter
25%
Dark Energy
70%
Dark Energy
+
Dark Matter
+
Seed Perturbations
(Inflation)
+
Baryo/Leptogenesis



No Big Bang
Supernovae
SNAP SN
Target
CMB
expands forever
recollapses eventually
Clusters
closed
flat
open
Ω_Λ
Ω_M
3
2
1
0
-1
0
1
2
3

Facts indicating finiteness of the universe in the past

1. Cosmological red shift
2. Lifetime of the old stars around13 billion years
3. Distant galaxies (galaxies far away) are poor in heavy elements
4. Fluctuations of CMB temperature (agreement with calculations based on13.7 billion years)
5. Possibility for stars to be born (substance not managed to spread uniformly)
6. Hydrogen is not fully processed into Helium

Open questions

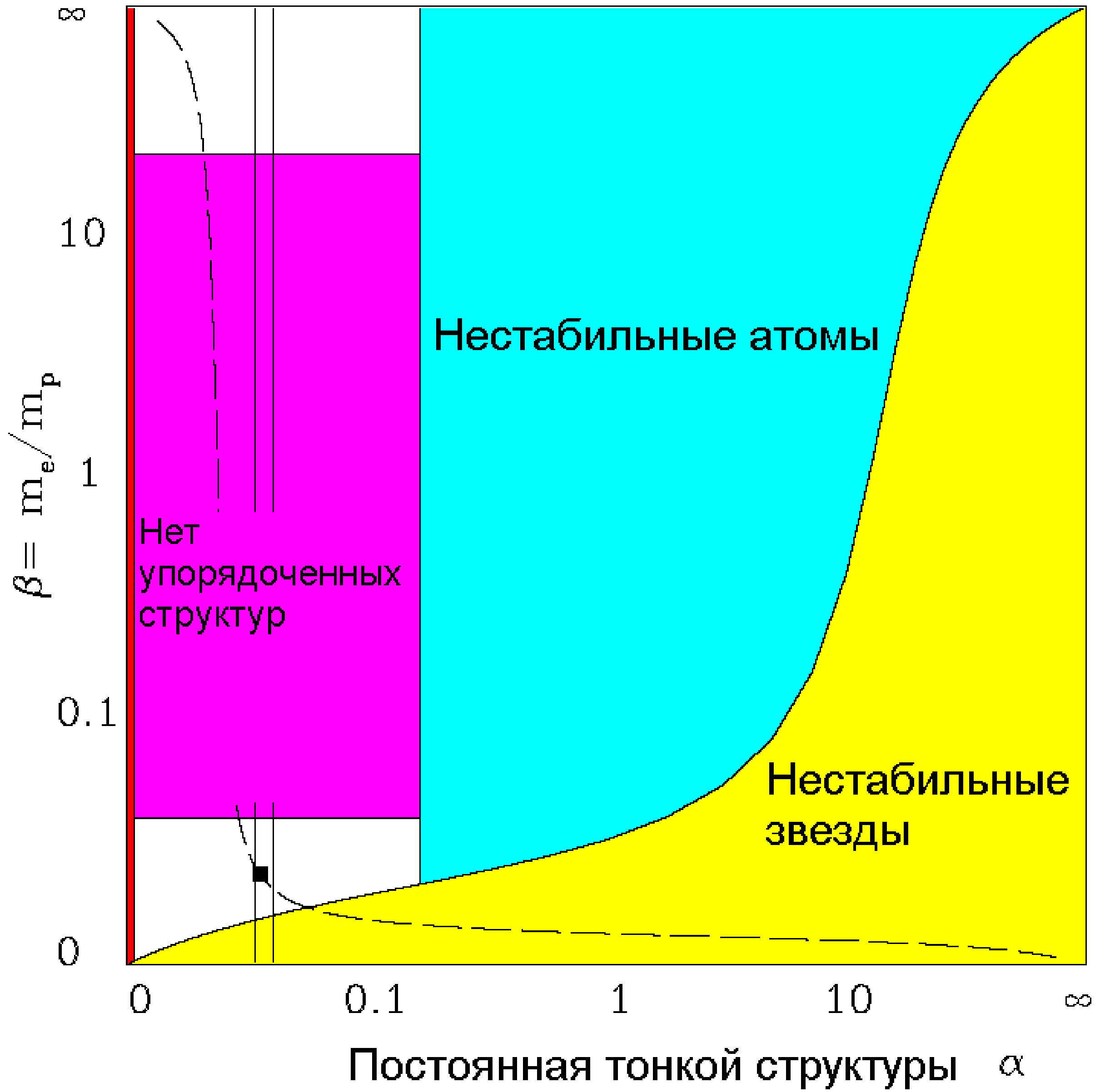
1. Weinberg-Salam model and Higgs mechanism
2. Symmetry breaking mechanism for particles and antiparticles
3. Early quasars and galaxies formation
4. The essence of dark matter and dark energy
5. Extra dimensions: stability, compactness, topology. Why there is only four dimensions in our world?
6. Origin of different sorts of particles.
7. Explanation for the value of quantities *c, h, G*
8. Unifying of the gravity and quantum theory
9. Arrow of time

Fine tuning of the parameters

Problems 2-7 are discussed in these lectures

Fine tuning of the Universe (FT)

- Small change in elementary particles parameters leads to impossibility for complex structures to form
- It is very unlikely to hit into the unbarred range
- Most likely there are different universes with all range of parameters



$$dl^2 = dx_1^2 + dx_2^2 + dx_3^2$$

$$ds^2 = c^2 dt^2 - (dx_1^2 + dx_2^2 + dx_3^2)$$

$$ds^2 = g_{ij} dx^i dx^j \,,\; i,\, j = 1, 2, 3, 4$$

$$ds^2 = g_{AB} dx^A dx^B \,,\; A, B = 1, 2, \ldots, D, D > 4.$$

$$ds^2 = g_{ij} dx^i dx^j + \phi(x)^{-1} h_{ab}(y) dx^a dx^b$$

Dynamic variables– 1

Universe should be described by dynamic variables

Dynamic variables of gravity is metric tensor $g_{\mu\nu}$

Coordinates dx^μ

Squared interval $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$

Minkowski space metric (space without rotations and accelerations)

$$\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$$

In curved space, one can always find a reference system in which

$$g_{\mu\nu} \to \eta_{\mu\nu}$$

Generally the metric tensor is very complex

$$g_{\mu\nu}\left(t, \vec{r}\right)$$

Dynamic variables – 2

On the scale of 100 Mpc the Universe is homogeneous, so it can be considered a perfect fluid with 4-speed

$$u^{\mu} = \frac{dr^{\mu}}{ds}$$

For such fluid one can use known hydrodynamic expression for the energy-momentum tensor

$$T_{\mu}^{\nu} = \left(p + \rho \right) u_{\mu} u^{\nu} - p \delta_{\mu}^{\nu}$$

Here we introduce the fluid pressure p and energy density ρ

In the stationary reference system 4-speed $u^{\mu} = (1,0,0,0)$

energy-momentum tensor $T_{\mu\nu} = \mathrm{diag}\left(\rho, -p, -p, -p\right)$

Dynamic variables – 3

- To describe the inflation usually we introduce scalar fields $\varphi(t,\vec{r})$
- In the curved space Lagrangian for these fields has the form

$$L_s = \frac{1}{2} g_{\mu\nu} \partial^{\mu}\varphi \partial^{\nu}\varphi - V(\varphi)$$

- Gravity affects the dynamics of the metric tensor

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{16\pi G} R - \Lambda \right) + S_m \qquad \delta S = 0$$

$$S_m = \int L_m \sqrt{g} d^4x$$

Lecture 2

Questions on lecture1

1. Express light year in inches
2. What are the Planck mass, size, time?
3. What is the size of the visible universe?
4. What are the dynamic variables in gravity?
5. What is the Riemann tensor??

Gravity. basic equations– 1

- General action:

$$S = \int \sqrt{-g}\,\frac{R}{2\kappa} d^4x + \int \sqrt{-g} L_m d^4x$$

$$\delta A^k = A^n dx^m dx^l R^k_{nml}$$

$$R = g^{\mu\nu} R_{\mu\nu}, \quad \kappa = \frac{8\pi G}{c^4}$$

- Here L_m – Lagrangian of matter, R – scalar Ricci, κ – gravitational constant,

$$g = \det g_{\mu\nu}$$

- Riemann tensor has a complex dependence on the metric tensor through the Christoffel symbols

$$\Gamma^i_{kl}$$

$$R_{ik} = \frac{\partial \Gamma^l_{ik}}{\partial x^l} - \frac{\partial \Gamma^l_{il}}{\partial x^k} + \Gamma^l_{ik}\Gamma^m_{lm} - \Gamma^m_{il}\Gamma^l_{km}$$

$$\Gamma^i_{kl} = \frac{1}{2} g^{im} \left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m} \right)$$

Gravity. basic equations – 2

- variation of the action *S*

$$\frac{\delta S}{\delta g_{\mu\nu}} = 0$$

- leads to the Hilbert-Einstein equation (H-E)

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu}$$

- Here energy-momentum tensor (EMT) is defined as follows

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}}$$

- The equations of motion can be obtained from

$$\nabla_\alpha T^\alpha_\mu = 0$$

Friedmann-Robertson-Walker metric – 1

Consider the case of a homogeneous and isotropic medium, when

$$T_{\mu\nu} = T_{\mu\nu}\left(t\right)$$

Then we can show that the metric tensor depends on one function - the scale factor *a*(*t*)

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} = dt^2 - a^2\left(t\right)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right]$$

$$d\Omega^2 = \sin^2\theta \, d\varphi^2 + d\theta^2$$

The parameter k can take the values -1, 0, 1, which corresponds to the open (hyperboloid), flat and closed universe

Problem reduces to finding the scale factor a(t), associated with a metric space and, consequently, the distances between objects.

Friedmann-Robertson-Walker metric – 2

Suppose that the environment is a perfect fluid with energy-momentum tensor

$$T_{\mu\nu} = \mathrm{diag}\left(\rho, p, p, p\right)$$

- From H-E equation we get

$$\begin{cases} 3\dfrac{\ddot{a}}{a} = -4\pi G\left(\rho + 3p\right) + \Lambda & - \quad (ii) \\ \left(\dfrac{\dot{a}}{a}\right)^2 + \dfrac{k}{a^2} = \dfrac{8\pi G}{3}\rho + \dfrac{\Lambda}{3} & - \quad (00) \end{cases}$$

Friedmann-Robertson-Walker metric – 3

We simplify the equation obtained as follows. We put into

$$\ddot{a} = \frac{1}{2}\frac{d\dot{a}^2}{da}$$

Equations (00) and (ii) from the H-E system. After simple calculations, we obtain the equation

$$\frac{d\rho}{da} = -\frac{3}{a}(p+\rho) \Leftrightarrow \frac{d\rho}{dt} = -3H(p+\rho)$$

- Both forms are important. In the last record Hubble parameter was entered

$$\boxed{H = \frac{\dot{a}}{a}}$$

Friedmann-Robertson-Walker metric – 4

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi G}{3}\rho + \Lambda/3,$$
$$\frac{d\rho}{da} + \frac{3(\rho + p)}{a} = 0.$$

Almost always $\frac{k}{a^2}$ <<1

$+ \quad p = w\rho$

We have two equations and three variables : *a*, *p*, *ρ*
We need to clarify what substance is under consideration.
To do this one should write the equation of state
For dust *w* = 0, for ultrarelativistic gas and photons *w* = 1/3

$$\frac{d\rho}{da} = -\frac{3}{a}(w+1)\rho$$

$$\rho = \frac{C}{a^{3(w+1)}}$$

Now we know how changes the density with increasing scale factor

Friedmann-Robertson-Walker metric– 5

- Find an expression for the scale factor

- We start from the equation

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3}$$

- Influence of dark energy is still considered unimportant; curvature of space is important only at the initial time, and after – negligible

$$\Lambda = 0, \quad \frac{k}{a^2} << 1$$

- Thus, the following equation is transformed into

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho = \frac{8\pi G}{3}\frac{C}{a^{3\gamma}}$$

Friedmann-Robertson-Walker metric – 6

- The solution of this equation

$$a(t) = a_{in}\left[1+\frac{2}{3\gamma}\sqrt{\frac{8\pi G}{3}\rho_{in}}\left(t-t_{in}\right)\right]^{\frac{2}{3\gamma}} \propto t^{\frac{2}{3(w+1)}}$$

$$a(t) \simeq a_{in}\left[\frac{2}{3\gamma}H_{in}\cdot(t-t_{in})\right]^{2/3\gamma}$$

$$H \equiv \frac{\dot{a}}{a} = \sqrt{\frac{8\pi G}{3}\rho}$$

- Thus, the radiation-dominant stage (RD, t > 10^{-27} s)

$$p = \rho/3 \qquad a(t) \propto \sqrt{t}$$

- For matter-dominant stage (dust, MD, t > 10000 – 30000 years)

$$p=0 \qquad a(t) \propto t^{\frac{2}{3}}$$

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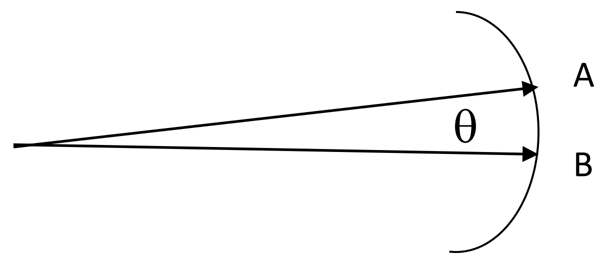
p=-ρ (w=-1) --- de Sitter stage

The physical meaning of the scale factor dependence on time

$$ds^2 = dt^2 - a(t)^2 \left[\frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2\theta \; d\varphi^2) \right]$$

r – dimensionless tags, a(t) - is dimensional

What is the distance from A to B, visible from the center and expressed in cm?



$$S = 4\pi \left[a(t) r \right]^2 \longrightarrow R(t) \equiv a(t) r \qquad \Delta l = R(t)\theta$$

Comoving and physical coordinates and distances.
(Points to the rubber surface, rulers are physical and attached to the space)

If A and B - standard light emitters, we can find the distance to the center of the redshift. To determine the distance AB it is necessary to measure an angle.

Scale factor– 1

- square of the interval

$$ds^2 = dt^2 - \frac{a^2(t)}{1-kr^2} dr^2$$

- Distance traveled by light determines causally related area.
- This distance is determined from the condition *ds* = 0
- However, the very notion of distance in cosmology is very convenient to define in different reference systems, depending on the task: it can be defined in dimensional physical coordinates or in dimensionless - comoving
- The calculation result is strongly dependent on the reference system!

Distance measured simultaneously by chain of observers

$$R_{inst} = a(t)\int_0^r \frac{dr}{\sqrt{1-kr^2}}$$

$$R_{inst} = \begin{cases} a(t)\,\mathrm{arcsin}(r), & k=1 \\ a(t)r, & k=0, \\ a(t) arcsh(r), & k=-1 \end{cases}$$

Distance traveled by light $\qquad ds = 0$

$$\int_0^t \frac{dt}{a(t)} = \int_0^r \frac{dr}{\sqrt{1-kr^2}}.$$

$$R(t)_{hor} \equiv a(t)r = \begin{cases} a(t)\sin\left(\int_0^t dt/a(t)\right) & k=1; \\ a(t)\int_0^t dt/a(t), & k=0 \\ a(t)sh\left(\int_0^t dt/a(t)\right), & k=-1. \end{cases}$$

Scale factor– 3

- The distance in comoving coordinates is dimensionless, it is like a rubber ruler changes its scale depending on the curvature of space
- scale factor is the dimensional one

- Distance in the physical system (dimensional)

$$R_{phys} = a(t)r = \begin{cases} a(t)\sin\left(\int_0^t \frac{dt'}{a(t')}\right), & k=1 \\ a(t)\int_0^t \frac{dt'}{a(t')}, & k=0 \\ a(t)\mathrm{sh}\left(\int_0^t \frac{dt'}{a(t')}\right), & k=-1 \end{cases}$$

Lecture 3

Questions on lecture 2

1. Connection of vectors with upper and lower indices
2. Write invariant volume element
3. Write Hilbert-Einstein action
4. What is the gravitational redshift
5. Energy - momentum tensor of an ideal fluid

De Sitter space – 1

ρ=0, Λ#0

- Consequence of the H-E equation with the FRW metric and ρ = 0

$$\dot{a}^2 - H_0^2 a^2 = -k$$

$$\Lambda \neq 0$$

$$H_0 = \sqrt{\frac{\Lambda}{3}}$$

$$\Lambda = 8\pi G \rho_0$$

- The solution of this equation for different values of *k*

$$a(t) = \begin{cases} H_0^{-1}\,\mathrm{ch}\left(H_0 t + C_1\right), & k = 1 \\ C_2 \exp\left(H_0 t\right), & k = 0 \\ H_0^{-1}\,\mathrm{sh}\left(H_0 t + C_3\right), & k = -1 \end{cases}$$

usually $\quad C_1 = C_3 = 0, \quad C_2 = H_0^{-1}$

De Sitter space – 2

- At large times *t* and *k* = 1 or *k* = -1 $\quad a(t) \sim \exp(H_0 t)$
- The difference is only for short times, because we consider t large enough to neglect the differences

$$a(t) = H_0^{-1}\exp(H_0 t)$$

- We define the distance traveled by light in comoving coordinates

$$ds^2 = dt^2 - a^2(t)\,dr^2 \equiv 0$$

$$dr = \frac{dt}{a(t)}$$

$$r(t,t') = \int_t^{t'} \frac{dt''}{a(t'')} = e^{-H_0 t} - e^{-H_0 t'} = L_{comov}(t,t')$$

De Sitter space – 3

- What is the distance light will pass for an infinitely long time?

$$L_{hor} = r\left(t, t' = \infty\right) = \exp\left(-H_0 t\right)$$

For an infinite time light passes a finite distance. And the later emitted light, the smaller the size of the horizon. Causally connected region decreases.

In comoving coordinates the horizon radius changes, rather than the space itself

- In physical coordinates

$$R = a\left(t\right) r$$

- Horizon size in these coordinates

$$R\left(t, t'\right) = H_0^{-1}\left(e^{H_0\left(t'-t\right)} - 1\right)$$

$$L_{phys,hor} = R\left(t, t' \to \infty\right) \to \infty$$

Inflation – 1

- Inflation - an exponentially rapid expansion of the universe
- For Inflation to arose one need gravity and the scalar field

$$L = \sqrt{-g}\left\{\frac{1}{16\pi G}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V(\varphi)\right\}$$

- What could be the fluctuation?

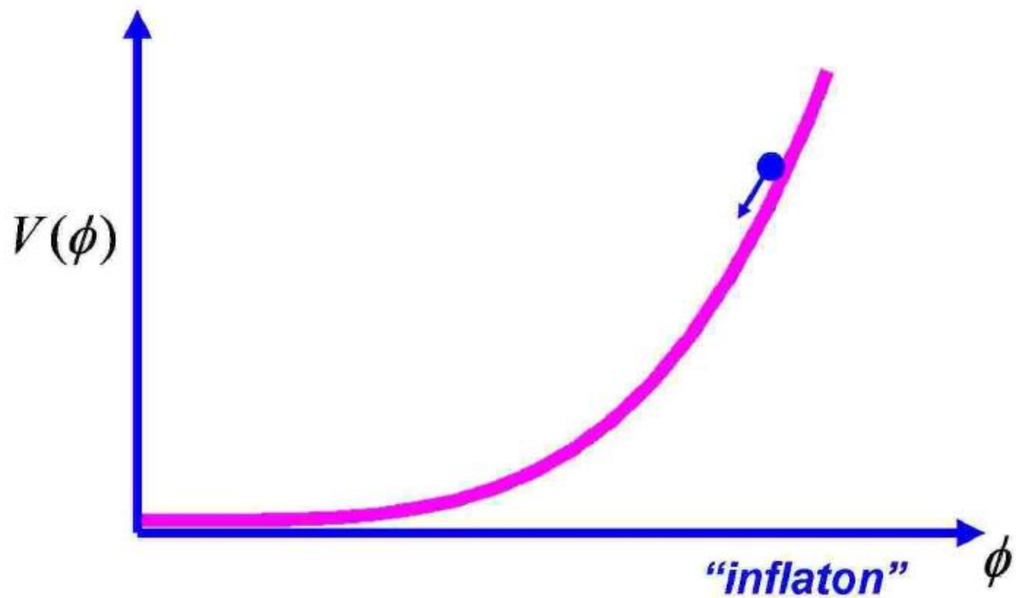
$$\begin{matrix} \Delta E \Delta t = \Delta E \Delta l \sim 1 \\ \Delta E \sim \varepsilon \Delta l^3 \end{matrix} \longrightarrow \Delta L \sim \varepsilon^{-1/4}$$

Modern value (in planck units): $\varepsilon \approx 10^{-12}$

$$\Delta L \sim \varepsilon^{-1/4} \sim 10^3 l_{Pl}$$

Inflation – 2

- All the time around us, fluctuations appear with $E \sim 10^{-3}\, M_{Pl}$ *and size* $\Delta l \sim 10^3\, l_{Pl}$
- To the casual observer, the lifetime of such fluctuations is negligible, but because its size is much larger than l_{Pl}, one can use Einstein's equations to describe the processes from the point of view of the internal observer



$$\partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu \varphi + \sqrt{-g} V'(\varphi) = 0$$

Inflation – 3

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi G}{3}\rho + \Lambda/3,$$

$$\rho = \frac{1}{2}\dot{\varphi}^2 + V(\varphi)$$

a(t), φ(t) – homogeneous distribution

$$\ddot{\varphi} + 3H\dot{\varphi} + V'(\varphi) = 0,$$

Friction can be big

$$H^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right)$$

$$H \equiv \frac{\dot{a}}{a}$$

Slow-roll approximation

$$3H\dot{\varphi} >> \ddot{\varphi}$$

$$3H\dot{\varphi} + V'(\varphi) \simeq 0$$

$$V'(\varphi) >> \ddot{\varphi}$$

$$\dot{\varphi}^2 << V(\varphi)$$

Inflation – 4

So:

- An important condition for the implementation of inflation is the slow change of the scalar field, i.e.

$$\frac{1}{2}\dot{\varphi}^2 \ll V(\varphi) \Leftrightarrow \dot{\varphi}\ddot{\varphi} \ll V'(\varphi)\dot{\varphi} \Leftrightarrow \ddot{\varphi} \ll V'(\varphi)$$

- Slow motion is carried, if the term responsible for friction is big

$$3H\left|\dot{\varphi}\right| \gg \left|\ddot{\varphi}\right|$$

- This allows you to further simplify the system of equations

$$\begin{cases} 3H\dot{\varphi} + V'(\varphi) \simeq 0 \\ H = \dfrac{\dot{a}}{a} = \sqrt{\dfrac{8\pi G}{3}V(\varphi)} \end{cases}$$

- Solution $\qquad a(t) \propto \exp(Ht)$

$$H \equiv \frac{\dot{a}}{a} \simeq \sqrt{\frac{8\pi G}{3} V(\varphi)}$$

$$V \simeq Const$$

$$H \simeq Const \longrightarrow a(t) \propto \exp(Ht)$$ Almost de Sitter

$$a(t) = \exp\left[\int_{t_{in}}^{t} H(\varphi) dt\right]$$

$$N \equiv \ln\left[\frac{a(t)}{a(t_{in})}\right] = \int_{t_{in}}^{t} H dt = \int_{\varphi_{in}}^{\varphi} H \frac{d\varphi}{\dot{\varphi}} = -\int_{\varphi_{in}}^{\varphi} \frac{3H(\varphi)^2 d\varphi}{V'(\varphi)}$$

$N_{end} \approx 60$

Inflation – 6

Define more accurately the condition of the slow roll

- For this purpose let us represent equation as follows

$$\dot{\varphi} = -\frac{V'(\varphi)}{3H}$$

- Express the second derivative

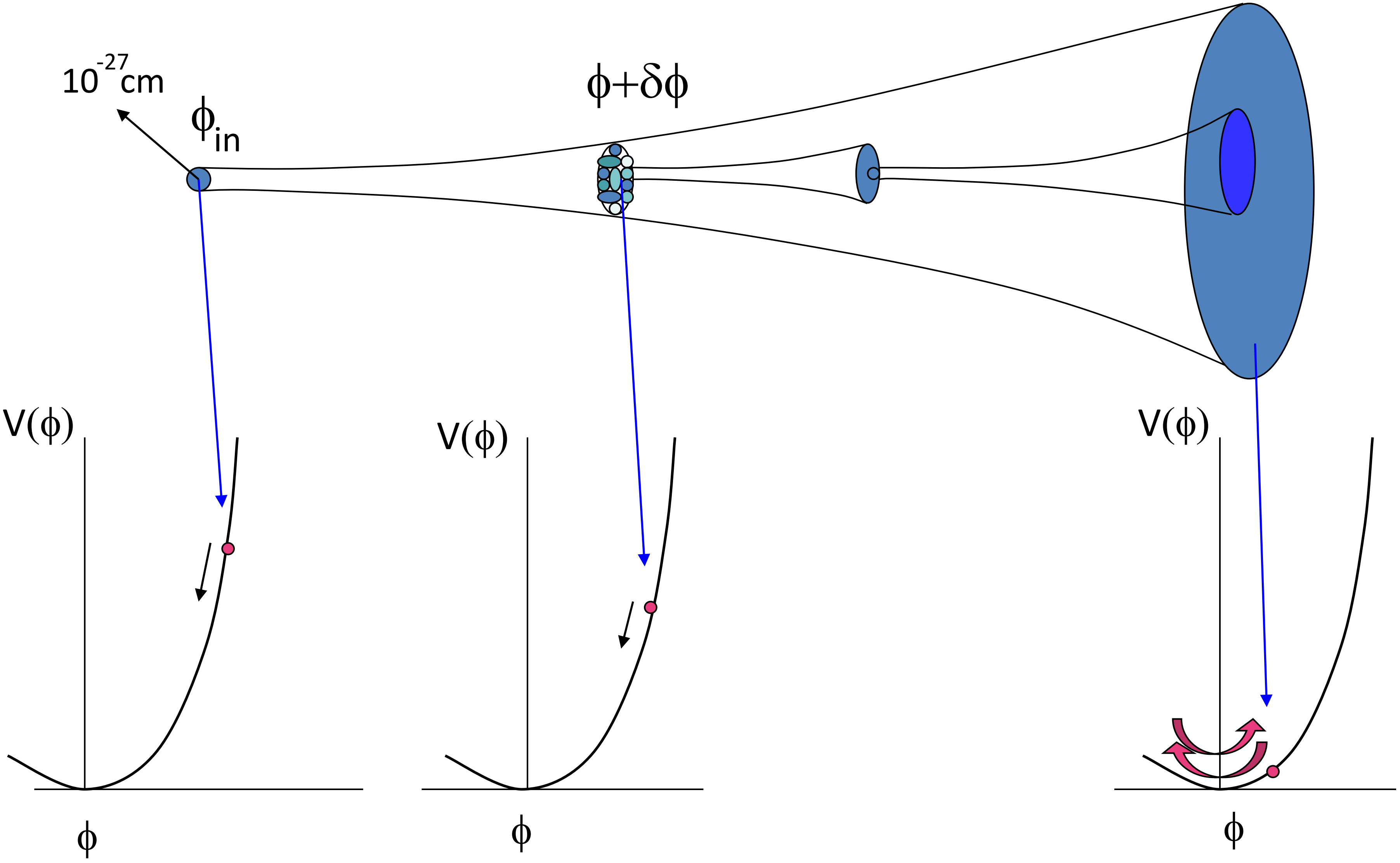
$$\ddot{\varphi} = \frac{V'(\varphi)}{3H^2(\varphi)} H'(\varphi)\dot{\varphi} - \frac{V''(\varphi)}{3H(\varphi)}\dot{\varphi}$$

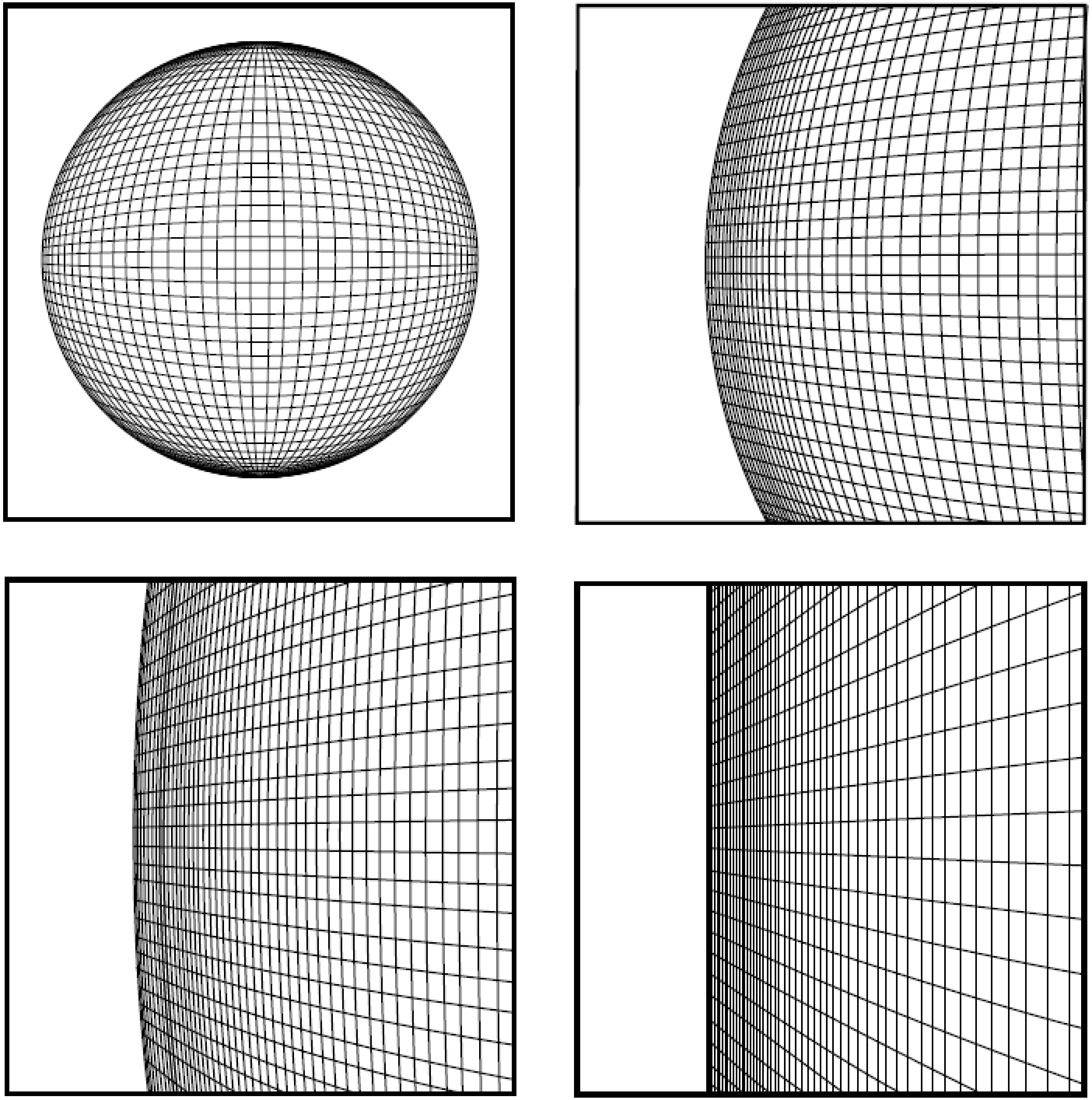
- Each of the terms must be small compared to $3H\dot{\varphi}$
- Using the equation for the Hubble parameter and the dynamics of the scale factor, we obtain

$$H = \frac{\dot{a}}{a} = \sqrt{\frac{8\pi G}{3} V(\varphi)}$$

$$\varepsilon \equiv \frac{M_{Pl}^2}{16\pi} \frac{V'^2(\varphi)}{V^2(\varphi)} \ll 1, \quad \eta \equiv \frac{M_{Pl}^2}{8\pi} \left| \frac{V''(\varphi)}{V(\varphi)} \right| \ll 1$$

Space and field alternation during inflation





Lecture 4

Questions on lecture 3

1. Scale factor
2. What is Hubble parameter
3. Connection between energy density with the scale factor
4. physical and comoving coordinates
5. The path traveled by light in a time t

Inflation – 8

EMT scalar field:

$$T_{\mu\nu} = \partial_\mu \varphi \partial_\nu \varphi - g_{\mu\nu} L$$

In a period of inflation, the condition satisfied

$$\frac{1}{2}\dot{\varphi}^2 \ll V(\varphi)$$

We neglect kinetic term in the Lagrangian

$$L = -V(\varphi)$$

Then

$$T_{\mu\nu} = \mathrm{diag}\left(V(\varphi), -V(\varphi), -V(\varphi), -V(\varphi)\right)$$

But for ideal fluid

$$T_{\mu\nu} = \mathrm{diag}\left(\rho, -p, -p, -p\right)$$

We get $\rho = V(\varphi)$ and pressure $p = -V(\varphi) < 0$