FUNDAMENTAL INTERACTIONS

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Dynamical variables – set of parameters that define behavior of the system depending on time.

Examples:
Temperature of the system in the equilibrium process T(t)
Coordinate of the material point

$$\vec{x}(t)$$

More complicated object – field: a set of quantities defined in some points in space and depending on time

Temperature field $$T_x(t) = T(x,t)$$

Electric field $$\vec{E}(x,t)$$

$$\varphi_x(t) = \varphi(x,t)$$

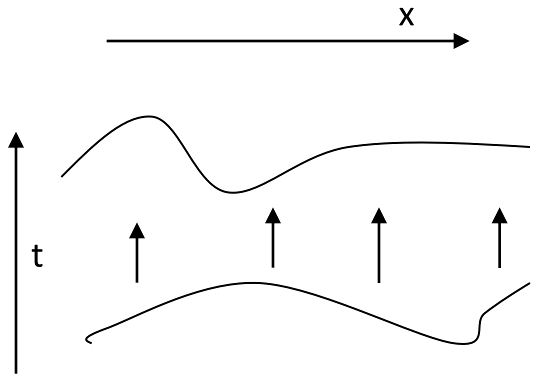
q(t) – generic dynamical variable

Fields are more fundamental objects. Oscillations of the fields near the stationary state can be seen as particles.
There are another states of the field – filed objects that have energy.

Notion of the energy density
Gravitational field is an exceptional case – its energy density can be chosen =0. BUT ONLY LOCALLY!
Scalar and vector fields

Configurations of the fields – Coulomb, black hole, field of a magnet.

Description of the field dynamics – Action → Minimum of the action→ Equations of motion



$$S = \int d^4 x L(\varphi, \partial_\mu \varphi),\ L(\varphi, \partial_\mu \varphi) = \frac{1}{2}(\partial_\mu \varphi)^2 - V(\varphi)$$

$$\frac{\delta S}{\delta \varphi} = 0 = \frac{\partial L}{\partial \varphi} - \partial_\mu \left(\frac{\partial L}{\partial \varphi_{,\mu}} \right)$$

How to find out whether an object is a particle?

$$\partial_\mu^2 \varphi = -V'_\varphi$$ ← Prove it – Exercise in the end of the lecture

Notion of a "particle" – object with a well-known connection between energy and momentum

$$E^2 = p^2 + m^2$$

What is the Lagrangian describing the particle?
What equation does the field describing the particle obey?

$$L = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{1}{2}m^2\phi^2 - V(\phi); \quad V(\phi) = \ldots a_n \phi^n \ldots$$

$$\partial_\mu \partial_\mu = \partial_0^2 - \partial_n^2$$

Equation of a free field. Plane waves – particles – oscillations of the field Φ(t,x)

$$\frac{\delta S}{\delta \varphi} = 0 = \frac{\partial L}{\partial \varphi} - \partial_\mu \left(\frac{\partial L}{\partial \varphi_{,\mu}}\right) \longrightarrow \begin{array}{l} \partial_\mu^2 \varphi + m^2 \varphi = 0; \quad \varphi(x,t) = Ce^{-iEt+ipx} \\ -E^2 + p^2 + m^2 = 0 \end{array}$$

– Exercise in the end of the lecture

Mass is a parameter before the quadratic term in the Lagrangian

Plane wave ⟷ flux of particles with certain energy
Orders of the field higher than two don't give plane waves and stationary fluxes and interpreted as interactions of the plane waves.

$$\partial_\mu^2 \varphi + m^2 \varphi = -na\varphi^{n-1}; \quad \varphi(x,t) \neq Ce^{-iEt+ipx}$$

Quantum physics

Set of the main postulates:

a. Rules for quantization (p→∂)

b. Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t}=H\Psi,\quad \Psi=const\cdot A$$

!

$$\Psi(x,t)=Ne^{iHt}\Psi(x,0)\rightarrow\Psi(x,t)=\int K(x,t\,|\,y,0)\Psi(y,0)dy \qquad (*)$$

We are interested in the amplitude of the transition from the point «a» in a moment of time t=0 to the point «b» in a moment t. In transitional moments of time particle can be situated in any point. That is the trajectory is random. This fact indicates completely different way to treat the quantum behavior of the system.

Let us assume:

$$K(x_2,t_2;x_1,t_1)=\int_{x_1}^{x_2}Dy\cdot e^{iS[y(t)]}$$

$$S=\int_{t_1}^{t_2}dtL(y,\partial_t y),\quad L(y,\partial_t y)=\frac{1}{2}(\partial_t y)^2-V(y)$$

Quantum mechanics and path integrals, R. Feynman, A. Hibbs

$$\Psi(x,t) = Ne^{iHt}\Psi(x,0) \to \Psi(x,t) = \int_{-\infty}^{\infty} K(x,t\,|\,y,0)\Psi(y,0)dy$$

$$K(x_2,t_2;x_1,t_1) = \int_{x_1}^{x_2} Dy \cdot e^{\frac{i}{\hbar}S[y(t)]}$$

Let us show that (*) obeys the Schrödinger equation

$$\Psi(x,t+\varepsilon) = \int_{-\infty}^{\infty} K(x,t+\varepsilon\,|\,y,t)\Psi(y,t)dy$$

$$\Psi(x,t+\varepsilon) = \int_{-\infty}^{\infty} \exp\left[\frac{i}{\hbar}\frac{m(x-y)^2}{2\varepsilon} - \frac{i}{\hbar}\varepsilon V\left(\frac{x+y}{2}\right)\right]\Psi(y,t)dz =$$

$$= \int_{-\infty}^{\infty} \exp\left[\frac{i}{\hbar}\frac{mz^2}{2\varepsilon} - \frac{i}{\hbar}\varepsilon V\left(x+\frac{z}{2}\right)\right]\Psi(x+z,t)dy$$

b. Amplitude of the transition

$$A(a\to b) = \left\langle b(t)\middle| e^{iHt} \middle| a(0)\right\rangle$$

$$= \int dx \Psi_b^*(x,t)e^{iHt}\Psi_a(x,0)$$

$$\Psi(x,t)+\varepsilon\frac{\partial\Psi}{\partial t}=\int_{-\infty}^{\infty}\exp\left[\frac{i}{\hbar}\frac{mz^2}{2\varepsilon}\right]\left[1-\frac{i}{\hbar}\varepsilon V\left(x+\frac{z}{2}\right)\right]\left[\Psi(x,t)+z\frac{\partial\Psi}{\partial x}+\frac{z^2}{2}\frac{\partial^2\Psi}{\partial x^2}\right]dz$$

$$\varepsilon\frac{\partial\Psi}{\partial t}=-\frac{i\varepsilon}{\hbar}V\Psi-\frac{\hbar\varepsilon}{2im}\frac{\partial^2\Psi}{\partial x^2}$$

$$i\hbar\frac{\partial\Psi}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}+V\Psi$$

Connection between the classical and quantum descriptions

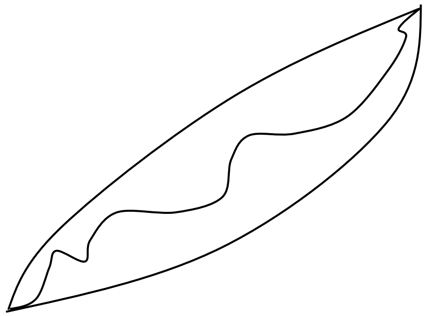
We need to have the amplitude of the transition

$$A(a \to b) = \langle b(t) | a(t) \rangle = \int dx \Psi_b^*(x,t) \Psi_a(x,t)$$

$$\left. \begin{array}{l} \Psi_a(x,0) \to \Psi_a(x,t) \\ |a(0)\rangle \quad \Rightarrow \quad |a(t)\rangle \end{array} \right|$$

Continual integral

$$A(a \to b) = \int_a^b Dq \cdot e^{iS[q(t)]/\hbar}$$



Consider the vacuum with $q_{cl} = const = 0$ e.g. electromagnetic wave in the vacuum.

Another extreme case: $S(q_{cl})/\hbar >> 1$

Trajectories corresponding to the minimum of the action give the dominant contribution to the integral

$$\left.\frac{\delta S(q)}{\delta q}\right|_{q_{cl}} = 0 \to q_{cl} \to S(q_{cl}) \to A_{ab} \sim e^{iS(q_{cl})/\hbar}; \quad S(q_{cl})/\hbar >> 1$$

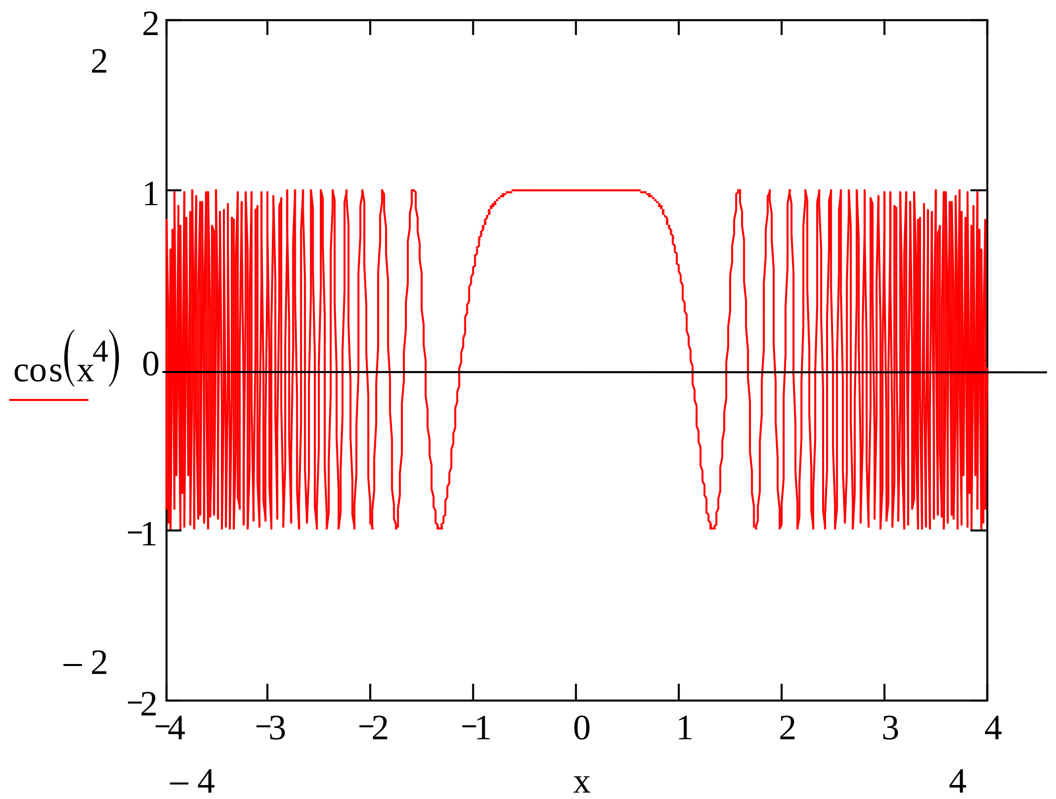
provement

$$I = \int dx e^{iS(x)} \approx e^{iS(x_{cl})}; \quad \left.\frac{\delta S}{\delta x}\right|_{x=x_{cl}} = 0$$

$$I = \int dx e^{iS(x)} = \int dx[\cos S(x) + i \sin S(x)]$$

$$S(x) = (x-a)^4; \quad x_{cl} = a \to S(x_{cl}) = 0; \to I \approx e^{iS(x_{cl})} = 1;$$

One can estimate I from the following plot



Main conclusions

1. Quadratic term describes the motion of a free particle

2. Terms $y^n, n > 2$ describe the interaction

3. $$A_{ab} \propto e^{iS_{cl}/\hbar}$$

4. Classical trajectories give the main contribution to the action, dS/dq=0 – all classical physics follow from here

How to choose the action? Key observation: there should exist quantities that conserve: charge, energy…

It turns out that to satisfy this condition it is sufficient to have the symmetry of the action about certain transitions such as

$$\varphi' = T\varphi$$
$$S[T\varphi] = S[\varphi] \longrightarrow \frac{dQ}{dt} = 0$$

$$\varphi' = T_1\varphi; \quad \varphi'' = T_2\varphi' = T_2T_1\varphi = T_3\varphi$$

Linear operators form the group. That is the action is invariant under this group (group of symmetry).

Observational fact – notation of the equations doesn't depend on the reference frame. This is the property of scalars – by definition. So the action and Lagrangian have to be scalars. As a consequence fields cannot transform arbitrarily while the transitions to the other reference frame – laws for field transformations should be definite to allow the construction of scalars from them.

Operators --- matrices

$$\chi_\alpha(x) \xrightarrow{a} \chi'_\alpha(x) = \mathcal{R}_{\alpha\beta}(a)\chi_\beta(x)$$

$$\chi_\alpha(x) \xrightarrow{a} \chi'_\alpha(x) \xrightarrow{a'} \chi''_\alpha(x)$$

$$\chi_\alpha(x) \xrightarrow{a''} \chi''_\alpha(x) \ .$$

$$\mathcal{R}_{\alpha\beta}(a')\mathcal{R}_{\beta\gamma}(a) = \mathcal{R}_{\alpha\gamma}(a'')$$

Set of matrices have to be «closed under the multiplication»

Spontaneous symmetry breaking

Example 1 – rod that is compressed by the force less and larger than the critical value.

Example 2 – real scalar field – discrete degeneration

Example 3 – complex scalar field – continuum of the ground states

Example 4 – rest symmetry – potential with two scalar fields

What are the average values of the field in the above mentioned cases?

Symmetries in physics

Symmetries by the example of the scalar field

$$L(\varphi) = \partial_\mu \varphi^* \partial_\mu \varphi - V\left(|\varphi|^2\right)$$

Equations of motion: $$\frac{\delta S}{\delta \varphi} = 0 = \frac{\partial L}{\partial \varphi} - \partial_\mu \left(\frac{\partial L}{\partial \varphi_{,\mu}} \right) \quad \Bigg| \quad \frac{\delta S}{\delta \varphi^*} = 0$$

$$L(\varphi, \varphi^*) = \partial_\mu \varphi^* \partial_\mu \varphi - V\left(|\varphi|^2\right)$$

$$* \quad L(\varphi', \varphi^{*}{}') = L(\varphi, \varphi^*)$$

$$e^{i\alpha}$$ Form the group (you should verify this)

$$\varphi \to \varphi' = e^{i\alpha}\varphi, \; \varphi^* \to \varphi^{*}{}' = e^{-i\alpha}\varphi$$

Symmetry leads to the conservation of some quantities.
Consider the infinitesimal transformations $$e^{i\alpha}\varphi \cong \varphi + i\alpha\varphi$$

(*) $$L(\varphi,\varphi^*) = L(\varphi + i\alpha\varphi, \varphi^* - i\alpha\varphi^*)$$

$$\frac{\partial L}{\partial \varphi} i\alpha\varphi + \frac{\partial L}{\partial \varphi_{,\mu}} i\alpha\varphi_{,\mu} + c.c. = 0$$

$$\frac{\delta S}{\delta \varphi} = 0 = \frac{\partial L}{\partial \varphi} - \partial_\mu\left(\frac{\partial L}{\partial \varphi_{,\mu}}\right)$$

$$\partial_\mu\left(i\alpha\varphi\frac{\partial L}{\partial \varphi_{,\mu}} + c.c.\right) = 0$$

$$\partial_\mu j_\mu = 0,\; j_\mu = i\left(\varphi\partial_\mu\varphi^* - \varphi^*\partial_\mu\varphi\right)$$

$$Q = \int j_0 d^3x = const$$

Equation (*) leads to the conservation of charge with time

Control:

$$\partial_0 Q = \int d^3x \partial_0 j_0 = -\int d^3x \partial_i j_i = -\int_{S_\infty} ds_i j_i = 0$$

Equation (*) leads to the conservation of some quantity!

Writing the coefficient function on the basis of the Lagrangian. The beginning.

The propagator – particle is situated in the “outer field”

$$\partial^2\varphi + m^2\varphi = j \rightarrow \left(E^2 - \vec{p}^2 - m^2\right)\varphi_p = -j_p \rightarrow$$

$$\varphi_p = \frac{-j_p}{E^2 - \vec{p}^2 - m^2}$$

To do in the end

For a free particle $j_p = 0$

$$\varphi_p = C\delta(E^2 - \vec{p}^2 - m^2)$$

This calculation is not obligatory on the lecture

$$0 = \delta W = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \chi_\alpha} \delta\chi_\alpha + \frac{\partial \mathcal{L}}{\partial \partial_\mu \chi_\alpha} \delta \partial_\mu \chi_\alpha \right]$$
$$= \int d^4x \left[\left\{ \frac{\partial \mathcal{L}}{\partial \chi_\alpha} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \chi_\alpha} \right) \right\} \delta\chi_\alpha + \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial \partial_\mu \chi_\alpha} \delta\chi_\alpha \right] \right] .$$

The first term above in the curly brackets vanishes because of the Euler-Lagrange equations of motion. The second can be rewritten in terms of the generator matrices g_i, since

$$\delta\chi_\alpha = \chi'_\alpha - \chi_\alpha = i\ \delta a_i (g_i)_{\alpha\beta} \chi_\beta \ . \tag{19}$$

$$0 = \delta W = -\int d^4x \delta a_i \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial \partial_\mu \chi_\alpha} \frac{1}{i} (g_i)_{\alpha\beta} \chi_\beta \right] .$$

Since the parameters δa_i are independent, it follows that the currents

$$J_i^\mu(x) = \frac{\partial \mathcal{L}}{\partial \partial_\mu \chi_\alpha(x)} \frac{1}{i} (g_i)_{\alpha\beta} \chi_\beta(x) \ , \qquad Q_i = \int d^3x J_i^o(x)$$

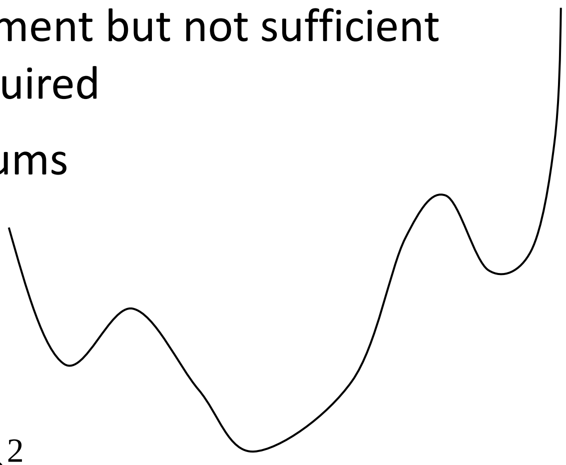
as a result of the symmetry, are conserved

$$\partial_\mu J_i^\mu(x) = 0 \ . \longrightarrow \frac{d}{dt} Q_i = 0 \ .$$

Spontaneous symmetry breaking

$$L(\varphi) = \frac{1}{2}\left(\partial_\mu \varphi\right)^2 + \mu^2\varphi^2 - \lambda\varphi^4 + const$$

$\varphi \to -\varphi$ Symmetry of interactions is the requirement but not sufficient
The symmetry of the ground state is required



Symmetry breaking – choose one of the possible vacuums

$$\varphi_m = \pm\frac{\mu}{\sqrt{2\lambda}}; \quad \varphi = \varphi_m + \phi$$

$$V(\varphi) = V(\varphi_m) + \left(6\lambda\varphi_m^2 - \mu^2\right)\phi^2 = V(\varphi_m) + 2\mu^2\phi^2$$

$$m_\phi = 2\mu$$

If there was an interaction with the other field then this field acquires the mass:

$$V_{\text{int}} = g\varphi\chi^2 = g\varphi_m\chi^2 + g\phi\chi^2;$$

$$m_\chi = \sqrt{2g\varphi_m}$$

Important: (the second order of) mass of the particle is proportional to the average value of the scalar field and to the coupling constant.

Spontaneous breaking of the global U(1) symmetry

The complex field ϕ . Lagrangian is symmetrical under the transformation

$$L(\phi) = \partial_i \phi^* + \mu^2 |\phi|^2 - \lambda |\phi|^4$$

$$\phi(x) \to \phi'(x) = e^{i\alpha}\phi(x); \quad \alpha = const$$

Minimum of the potential

$$|\phi(x)| = \phi_0 = \frac{\mu}{\sqrt{2\lambda}} e^{i\alpha}; \forall \alpha$$

Let us choose α=0

Fluctuations of the field can be written as

$$\phi(x) = \left(\phi_0 + \chi(x)\right) e^{i\theta(x)}$$

Then the Lagrangian in the lowest order

$$L(\phi(x)) \simeq \left(-i\partial_i\theta \cdot \phi_0 + \partial_i \chi\right)\left(i\partial_i\theta \cdot \phi_0 + \partial_i \chi\right) - V(\chi) =$$

$$= \left(\partial_i \chi\right)^2 + \left(\partial_i \theta\right)^2 \phi_0{}^2 - V(\chi)$$

As a consequence of spontaneous symmetry breaking we have the massless (Nambu-Goldstone) field θ

Questions about the previous lecture

1. Requirements for the theory to have the laws of conservations.
2. Why do fields transformations form the group?
3. Give the examples of symmetrical Lagrangian and group of symmetry.
4. What is the spontaneous symmetry breaking?
5. How the masses of particles are formed?